A New Fekete-Szego Inequality with Classesof Analytic Function Along with Its Subclasses, Extremals and Singularaties

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Abstract:

In this Paper we have introduced aNewFekete-Szegö inequality with classes of analytic functions along with its subclassesextremals and Singularities by using principle of subordination and as so obtained sharp upper Bound of the function.

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$belonging to these classes are also obtained.

Keywords:Bounded functions,Fekete-Szegöinequality,convex function, extremal function, Starlike functions, Univalent functions.

1.Introduction

Let $\mathcal{A}$denotes the class of the function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

Which are analytic function in the unit disc$\mathbb{E} = \{z: |z| < 1|\}$,

Let$\mathcal{S}$ be the class of the functions of the form (1) which are analytic univalent in$\mathbb{E}$. BieberBach[8] proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. And Löwner[5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.. With the known estimates this inequality plays an important role to determining estimates of higher coefficients for some sub classes of $\mathcal{S}$. {Chhichra[11], Babalola[6]}.

Using Löwner'smethod[5], Fekete and szego investigated a well known relation between a_3 and a_2^2 for the class

$|a_3 - \mu a_2^2| \leq$

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$$\begin{cases} 3-4\mu & ,if\,\mu\le 0 \\ 1+2e^{\left(\frac{-2\mu}{1-\mu}\right)} & ,if\ 0\le\mu\le 1 \\ 4\mu-3 & ,if\,\mu\ge 1 \end{cases} \quad (2)$$

The Fekete–Szegő inequality is an inequality for the coefficients of univalent analytic functions found by Fekete and Szegő , related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete–Szegő problem.

Let S^* be the subclass of $\mathcal{S}$ of univalent convex functions $h(z)=z+\sum_{n=2}^{\infty}c_n z^n\in\mathcal{A}$ satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)}>0, z\in\mathbb{E}. \qquad (3)$$

We are aware that a function $f(z)\in\mathcal{A}$ is said to be close to convex if there exist $g(z)\in S^*$ such that

$$Re\frac{((zf'(z))}{g(z)}>0, z\in\mathbb{E}. \qquad (4)$$

Kaplan[18] proved that close to convex functions are univalent.

$$S^*(A,B)=\{f(z)\in\mathcal{A};\ \frac{((zf'(z))}{g(z)}\prec\frac{1+Az}{1+Bz}, -1\le B\le A\le 1, z\in\mathbb{E}\} \qquad (5)$$

Where $S^*(A,B)$ is a subclass of S^*.

Fekete-Szegö problem was studied by Abedel-Gawad[4] in the context of alpha quasi-convex function. Goel and Mehrok[13], Al-Shaqsi and Darus[1], Hayami and Owa[17], Al-Abbadi and Darus[9] have investigated the upper bound of $|a_3-\mu a_2^2|$ for different functions in the class S.

Gurmeetsingh et al.[3] also introduced the class of inverse Starlike functions

$g(z)=z+\sum_{n=2}^{\infty}b_n z^n\in A$ which satisfies

$$Re\left(\frac{zf(z)}{2\int_0^z f(z)dz}\right)>0,\ z\in E \quad \text{i.e.}\ \frac{zf(z)}{2\int_0^z f(z)dz}<\frac{1+z}{1-z}$$

Gandhi et al.[11]and Rathore et al.[2]established a new class of analytic functions with Fekete-szego inequality using subordination method.
We introduce the class $\mathcal{A}(\alpha,\beta)$ of functions $g(z)=z+\sum_{n=2}^{\infty}b_n z^n\in\mathcal{A}$ with satisfying the condition

$$x\left[\frac{\left[z\{zf'(z)\}'\right]'}{f'(z)}\right]+y\left[\frac{z\{zf'(z)\}'}{f'(z)}\right]\prec\left(\frac{1+z}{1-z}\right) \qquad (6)$$

Let $\mathcal{A}(\alpha,\beta;A,B)$ denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z)=z+\sum_{n=2}^{\infty}b_n z^n\in\mathcal{A}$ satisfying the condition

$$x\left[\frac{\left[z\{zf'(z)\}'\right]'}{f'(z)}\right]+y\left[\frac{z\{zf'(z)\}'}{f'(z)}\right]\prec\left(\frac{1+Az}{1+Bz}\right);\ -1\le B\le A\le 1 \qquad (7)$$

Let $\mathcal{A}(\alpha,\beta;\delta)$ denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z)=z+\sum_{n=2}^{\infty}b_n z^n\in\mathcal{A}$ with satisfying the condition

$$\boldsymbol{x}\left[\frac{\left[z\{zf'(z)\}'\right]'}{f'(z)}\right]+\boldsymbol{y}\left[\frac{z\{zf'(z)\}'}{f'(z)}\right]\prec\left(\frac{1+z}{1-z}\right)^{\eta}\quad;\quad \eta>0 \tag{8}$$

Let $\mathcal{A}(\alpha,\beta;A,B,\delta)$denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{A}$with satisfying the condition

$$\boldsymbol{x}\left[\frac{\left[z\{zf'(z)\}'\right]'}{f'(z)}\right]+\boldsymbol{y}\left[\frac{z\{zf'(z)\}'}{f'(z)}\right]\prec\left(\frac{1+Az}{1+Bz}\right)^{\eta}-1\leq B\leq A\leq 1,\eta>0 \tag{9}$$

Here, Symbol $\prec$ stands for subordination.

Principle of Subordination : If $f(z)$ and $F(z)$ are two functions which are analytic in $\mathbb{E}$, then $f(z)$ is called a subordinate to $F(z)$ in $\mathbb{E}$, if there exists a function $w(z)$ which is analytic in $\mathbb{E}$ satisfying the conditions

(i)$w(0)=0$ and (ii) $|w(z)|<1$

such that $f(z)=F(w(z))$, where $z\in\mathbb{E}$ and we denote it as $f(z)\prec F(z)$. Let $\mathcal{U}$ denote the class of analytic bounded functions of the form

$$w(z)=\sum_{n=1}^{\infty}d_nz^n,\ w(0)=0,|w(z)|<1$$

Having the restrictions$|d_1|\leq 1,|d_2|\leq 1-|d_1|^2$.

2. Main Results :

THEOREM 1. : Prove that

$$|a_3-\mu a_2^2|\leq$$

$$\begin{cases}\dfrac{(3x+y+2)}{3(4x+y)(3x+y)}-\dfrac{\mu}{(3x+y)^2} & ,if\mu\leq\dfrac{2(3x+y)}{3(4x+y)} & (10)\\[2ex] \dfrac{1}{3(4x+y)} & ,if\dfrac{2(3x+y)}{3(4x+y)}\leq\mu\leq\dfrac{2(3x+y+1)(3x+y)}{3(4x+y)} & (11)\\[2ex] \dfrac{\mu}{(3x+y)^2}-\dfrac{(3x+y+2)}{3(4x+y)(3x+y)} & ,if\mu\geq\dfrac{2(3x+y+1)(3x+y)}{3(4x+y)} & (12)\end{cases}$$

the results aresharp.

Proof:

On Expanding (6) we have

$$x+y+(6x+2y)a_2z+(24ax_3+6ya_3-12xa_2^2-4ya_2^2)z^2\prec 1+2c_1z+2(c_2+c_1^2)z^2+.......$$

(13)

After identifying the terms in (13), we have

$$|a_3 - \mu a_2^2| \le \left| \frac{1}{6(4x+y)} \left\{ 2c_2 + 2c_1^2 + \frac{4c_1^2}{(3x+y)} \right\} - \mu \frac{c_1^2}{(3x+y)^2} \right|$$

This leads to

$$|a_3 - \mu a_2^2| \le \frac{1}{3(4x+y)} + \left[\left| \frac{(3x+y+2)}{3(4x+y)(3x+y)} - \frac{\mu}{(3x+y)^2} \right| - \frac{1}{3(3x+y)} \right] |c_1|^2 \quad (14)$$

Case I : when , $\mu \le \frac{(3x+y+2)(3x+y)}{3(4x+y)}$

$$|a_3 - \mu a_2^2| \le \frac{1}{3(4x+y)} + \left[\frac{2}{3(4x+y)(3x+y)} - \frac{\mu}{(3x+y)^2} \right] |c_1|^2 \quad (15)$$

Subcase I(a) : when , $\mu \le \frac{\mathbf{2}(3x+y)}{\mathbf{3}(4x+y)}$

$$|a_3 - \mu a_2^2| \le \frac{(3x+y+2)}{3(4x+y)(3x+y)} - \frac{\mu}{(3x+y)^2} \quad (16)$$

Subcase I(b) : when , $\mu \ge \frac{\mathbf{2}(3x+y)}{\mathbf{3}(4x+y)}$

$$|a_3 - \mu a_2^2| \le \frac{1}{3(4x+y)} \quad (17)$$

Case II : when , $\mu \ge \frac{(3x+y+2)(3x+y)}{3(4x+y)}$

$$|a_3 - \mu a_2^2| \le \frac{1}{3(4x+y)} + \left[\left| \frac{\mu}{(3x+y)^2} - \frac{(3x+y+2)}{3(4x+y)(3x+y)} \right| \right] \quad (18)$$

Subcase II(a) : when , $\mu \le \frac{2(3x+y+1)(3x+y)}{3(4x+y)}$

$$|a_3 - \mu a_2^2| \le \frac{1}{3(4x+y)} \quad (19)$$

Subcase II(b) : when , $\mu \ge \frac{2(3x+y+1)(3x+y)}{3(4x+y)}$

$$|a_3 - \mu a_2^2| \le \frac{\mu}{(3x+y)^2} - \frac{(3x+y+2)}{3(4x+y)(3x+y)} \quad (20)$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \le \frac{1}{3(4x+y)} \quad (21)$$

iff

$$\frac{\mathbf{2}(3x+y)}{\mathbf{3}(4x+y)} \le \mu \le \frac{2(3x+y+1)(3x+y)}{3(4x+y)}$$

Extremal function

Extreme value for first and third function is $z[1+pz]^q$

where p $=\frac{3(4x+y)-2(3x+y+2)(3x+y)}{\mathbf{3}(4x+y)(3x+y)}$, q = $\frac{\mathbf{3}(4x+y)}{3(4x+y)-2(3x+y+2)(3x+y)}$

Extreme value for second function is $\frac{z}{(1-z^2)^p}$

where p = $\frac{1}{\mathbf{3}(4x+y)}$

THEOREM 2. : Prove that

$$|a_3-\mu a_2^2| \le$$

$$\begin{cases} \frac{A-B}{6(4x+y)}\left[\frac{(A-B)-B(3x+y)}{(3x+y)}\right]-\frac{(A-B)^2\mu}{4(3x+y)^2} \ , if \mu \le \frac{\mathbf{2}}{\mathbf{3}}\left[\frac{(A-B)-B(3x+y)-3x-y}{(A-B)(4x+y)}\right](3x+y) & (22)\\ \frac{A-B}{6(4x+y)} \ , \quad if \frac{2}{3}\left[\frac{(A-B)-B(3x+y)-3x-y}{(A-B)(4x+y)}\right](3x+y) \le \mu \le \frac{2}{3}\left[\frac{(A-B)-B(3x+y)+3x+y}{(A-B)(4x+y)}\right](3x+y) & (23)\\ \frac{(A-B)^2\mu}{4(3x+y)^2}-\frac{A-B}{6(4x+y)}\left[\frac{(A-B)-B(3x+y)}{(3x+y)}\right] \ , if \mu \ge \frac{\mathbf{2}}{\mathbf{3}}\left[\frac{(A-B)-B(3x+y)+3x+y}{(A-B)(4x+y)}\right](3x+y) & (24) \end{cases}$$

the results are sharp.

Proof:

On Expanding (7) we have

$$x+y+(6x+2y)a_2z+(24ax_3+6ya_3-12xa_2^2-4ya_2^2)z^2 \prec 1+(A-B)c_1z+(A-B)(c_2-Bc_1^2)z^2+\dots \quad (25)$$

After identifying the terms in (25), we have

$$|a_3-\mu a_2^2| \le \left[\left|\frac{(A-B)(c_2-Bc_1^2)}{6(4x+y)}+\frac{(A-B)^2c_1^2}{3(4x+y)(3x+y)}-\frac{(A-B)^2\mu}{4(3x+y)^2}\right|\right]|c_1|^2$$

This leads to

$$|a_3-\mu a_2^2| \le \frac{A-B}{6(4x+y)}+\left[\left|\frac{(A-B)^2}{6(4x+y)(3x+y)}-\frac{B(A-B)}{6(4x+y)}-\frac{(A-B)^2\mu}{4(3x+y)^2}\right|-\frac{A-B}{6(4x+y)}\right]|c_1|^2 \quad (26)$$

Case I : when , $\mu \le \frac{\mathbf{2}}{\mathbf{3}}\left[\frac{(A-B)-B(3x+y)}{(A-B)(4x+y)}\right](3x+y)$

$$|a_3-\mu a_2^2| \le \frac{A-B}{6(4x+y)}+\left[\left|\frac{A-B}{6(4x+y)}\left[\frac{(A-B)-B(3x+y)-3x-y}{(3x+y)}\right]-\frac{(A-B)^2\mu}{4(3x+y)^2}\right|\right] \quad (27)$$

Subcase I(a) : when , $\mu \le \frac{\mathbf{2}}{\mathbf{3}}\left[\frac{(A-B)-B(3x+y)-3x-y}{(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \le \frac{A-B}{6(4x+y)}\left[\frac{(A-B)-B(3x+y)}{(4x+y)}\right] - \frac{(A-B)^2\mu}{4(3x+y)^2}$(28)

Subcase I(b) : when , $\mu \ge \frac{2}{3}\left[\frac{(A-B)-B(3x+y)-3x-y}{(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \le \frac{A-B}{6(4x+y)}$(29)

Case II :when , $\mu \ge \frac{2}{3}\left[\frac{(A-B)-B(3x+y)}{(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \le \frac{A-B}{6(4x+y)} + \left[\left|\frac{(A-B)^2\mu}{4(3x+y)^2} - \frac{A-B}{6(4x+y)}\left[\frac{(A-B)-B(3x+y)+3x+y}{(3x+y)}\right]\right|\right]$(30)

Subcase II(a) : when , $\mu \le \frac{2}{3}\left[\frac{(A-B)-B(3x+y)+3x+y}{(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \le \frac{A-B}{6(4x+y)}$(31)

Subcase II(b) : when , $\mu \ge \frac{2}{3}\left[\frac{(A-B)-B(3x+y)+3x+y}{(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \le \frac{(A-B)^2\mu}{4(3x+y)^2} - \frac{A-B}{6(4x+y)}\left[\frac{(A-B)-B(3x+y)}{(3x+y)}\right]$(32)

Combining subcase II(a) and subcase I(b), we get

$|a_3 - \mu a_2^2| \le \frac{A-B}{6(4x+y)}$(33)

iff,

$$\frac{2}{3}\left[\frac{(A-B)-B(3x+y)-3x-y}{(A-B)(4x+y)}\right](3x+y) \le \mu \le \frac{2}{3}\left[\frac{(A-B)-B(3x+y)+3x+y}{(A-B)(4x+y)}\right](3x+y)$$

Extremal function

Extreme value for first and third function is

$$z[1+pz]^q$$

where p = $\frac{3(A-B)(3x+y)-4\{(A-B)-B(3x+y)\}(3x+y)}{6(4x+y)(3x+y)}$, q = $\frac{3(4x+y)(A-B)}{3(A-B)(3x+y)-4\{(A-B)-B(3x+y)\}(3x+y)}$

Extreme value for second function is $\frac{z}{(1-z^2)^p}$

where p = $\frac{A-B}{6(4x+y)}$

Singularities:

Special caseson (33) when $A \ne B$

i) If A$> 0 , B > 0$ then this inequality is holds only for A>B.

ii) If A$> 0 , B < 0$ then this inequality is holds for all values of A and B.

iii)If A$< 0 , B < 0$ then this inequality is holds only for B>A.

iv) If A$< 0 , B > 0$ then this case does not valid.

THEOREM 3. : Prove that

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{(3x+y+2)\eta^2}{3(4x+y)(3x+y)} - \frac{\mu\eta^2}{(3x+y)^2} \quad , if \mu \leq \frac{\{\eta(3x+y+2)-3x-y\}(3x+y)}{3\eta(4x+y)} & (34) \\ \frac{\eta}{3(4x+y)} \quad , if \frac{\{\eta(3x+y+2)-3x-y\}(3x+y)}{3\eta(4x+y)} & (35) \\ \frac{\mu\eta^2}{(3x+y)^2} - \frac{(3x+y+2)\eta^2}{3(4x+y)(3x+y)} \quad , if \mu \geq \frac{\{\eta(3x+y+2)+3x+y\}(3x+y)}{3\eta(4x+y)} & (36) \end{cases}$$

the results are sharp.

Proof:

On Expanding (8) we have

$$x + y + (6x + 2y)a_2 z + (24ax_3 + 6ya_3 - 12xa_2^2 - 4ya_2^2)z^2 \prec 1 + 2\eta c_1 z + 2\eta(c_2 + \eta c_1^2)z^2 + \ldots$$

(37)

After identifying the terms in (37), we have

$$|a_3 - \mu a_2^2| \leq \left| \frac{1}{6(4x+y)} \left\{ 2\eta c_2 + 2\eta^2 c_1^2 + \frac{4\eta^2 c_1^2}{(3x+y)} \right\} - \mu \frac{\eta^2 c_1^2}{(3x+y)^2} \right|$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{\eta}{3(4x+y)} + \left[\left| \frac{(3x+y+2)\eta^2}{3(3x+y)(4x+y)} - \mu \frac{\eta^2 c_1^2}{(3x+y)^2} \right| - \frac{\eta}{3(4x+y)} \right] |c_1|^2 \quad (38)$$

Case I :when , $\mu \leq \frac{(3x+y+2)(3x+y)}{3(4x+y)}$

$$|a_3 - \mu a_2^2| \leq \frac{\eta}{3(4x+y)} + \left[\left| \frac{(3x+y+2)\eta^2}{3(3x+y)(4x+y)} - \frac{\eta}{3(4x+y)} - \mu \frac{\eta^2 c_1^2}{(3x+y)^2} \right| \right] |c_1|^2 \quad (39)$$

Subcase I(a) : when , $\mu \leq \frac{\{\eta(3x+y+2)-3x-y\}(3x+y)}{3\eta(4x+y)}$

$$|a_3 - \mu a_2^2| \leq \left[\frac{(3x+y+2)\eta^2}{3(4x+y)(3x+y)} - \mu \frac{\eta^2}{(3x+y)^2} \right] \quad (40)$$

Subcase I(b) : when , $\mu \geq \frac{\{\eta(3x+y+2)-3x-y\}(3x+y)}{3\eta(4x+y)}$

$|a_3 - \mu a_2^2| \leq \frac{\eta}{3(4x+y)}$(41)

Case II :when , $\mu \geq \frac{(3x+y+2)(3x+y)}{3(4x+y)}$

$$|a_3 - \mu a_2^2| \leq \frac{\eta}{3(4x+y)} + \left[\left|\mu \frac{\eta^2}{(3x+y)^2} - \frac{(3x+y+2)\eta^2}{3(3x+y)(4x+y)}\right| + \frac{\eta}{3(4x+y)}\right] |c_1|^2$$

$$|a_3 - \mu a_2^2| \leq \frac{\eta}{3(4x+y)} + \left[\left|\mu \frac{\eta^2}{(3x+y)^2} - \frac{\eta}{3(4x+y)} \left\{\frac{3(3x+y+2)+3x+y}{3(3x+y)}\right\}\right|\right] |c_1|^2 \quad (42)$$

Subcase II(a) : when , $\mu \leq \frac{\{\eta(3x+y+2)+3x+y\}(3x+y)}{3\eta(4x+y)}$

$|a_3 - \mu a_2^2| \leq \frac{\eta}{3(4x+y)}$(43)

Subcase II(b) : when , $\mu \geq \frac{\{\eta(3x+y+2)+3x+y\}(3x+y)}{3\eta(4x+y)}$

$|a_3 - \mu a_2^2| \leq \mu \frac{\eta^2}{(3x+y)^2} - \frac{(3x+y+2)\eta^2}{3(4x+y)(3x+y)}$(44)

Combining subcase II(a) and subcase I(b), we get

$|a_3 - \mu a_2^2| \leq \frac{\eta}{3(4x+y)}$(45)

iff

$$\frac{\{\eta(3x+y+2)-3x-y\}(3x+y)}{3\eta(4x+y)} \leq \mu \leq \frac{\{\eta(3x+y+2)+3x+y\}(3x+y)}{3\eta(4x+y)}$$

Extremal function

Extreme value for first and third function is $z[1+pz]^q$(46)

where $p = \frac{3\eta(4x+y)-2\eta(3x+y+2)(3x+y)}{3(4x+y)(3x+y)}$,$q = \frac{3\eta(4x+y)}{3\eta(4x+y)-2\eta(3x+y+2)(3x+y)}$

Extreme value for second function is $\frac{z}{(1-z^2)^p}$(47)

where $p = \frac{\eta}{3(4x+y)}$

Singularities:

Special cases on (45)

1. If $\eta > 0$ then the result is hold for all values of η.
2. If $\eta < 0$then the result is not valid.

Hence only (1) case is applicable on this theorem.

THEOREM 4. : Prove that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(A-B)\eta}{6(4x+y)}\left[\frac{(A-B)\eta - B(3x+y)}{(3x+y)}\right] - \frac{(A-B)^2\eta^2\mu}{4(3x+y)^2} & , if \mu \leq \frac{2(3x+y)[(A-B)\eta - B(3x+y) - 3x - y]}{3\eta(4x+y)(A-B)} \\ \frac{(A-B)\eta}{6(4x+y)} & , iff \frac{2(3x+y)[(A-B)\eta - B(3x+y) - 3x - y]}{3\eta(4x+y)(A-B)} \leq \mu \leq \frac{2(3x+y)[(A-B)\eta - B(3x+y) + 3x + y]}{3\eta(4x+y)(A-B)} \\ \frac{(A-B)^2\eta^2\mu}{4(3x+y)^2} - \frac{(A-B)\eta}{6(4x+y)}\left[\frac{(A-B)\eta - B(3x+y)}{(3x+y)}\right] & , if \mu \geq \frac{2(3x+y)[(A-B)\eta - B(3x+y) + 3x + y]}{3\eta(4x+y)(A-B)} \end{cases}$$

the results are sharp.

Proof:

On Expanding (9) we have

$$x + y + (6x + 2y)a_2 z + (24ax_3 + 6ya_3 - 12xa_2^2 - 4ya_2^2)z^2 \prec 1 + (A-B)c_1\eta z + (A-B)\eta(c_2 - B\eta c_1^2)z^2 + \ldots$$

(51)

After identifying the terms in (51), we have

$$|a_3 - \mu a_2^2| \leq \left[\left|\left\{\frac{(A-B)\eta}{6(4x+y)}(c_2 - B\eta c_1^2) + \frac{(A-B)^2\eta^2\mu c_1^2}{6(4x+y)(3x+y)}\right\} - \frac{(A-B)^2\mu\eta^2}{4(3x+y)^2}c_1^2\right|\right]$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\eta}{6(4x+y)} + \left[\left|\frac{(A-B)^2\eta^2}{6(4x+y)(3x+y)} - \frac{B(A-B)\eta}{6(4x+y)} - \frac{(A-B)^2\mu\eta^2}{4(3x+y)^2}\right| - \frac{(A-B)\eta}{6(4x+y)}\right]|c_1|^2 \quad (52)$$

Case I : when , $\mu \leq \frac{2}{3}\left[\frac{(A-B)\eta - B(3x+y)}{\eta(A-B)(4x+y)}\right](3x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\eta}{6(4x+y)} + \left[\left|\frac{(A-B)\eta}{6(4x+y)}\left[\frac{(A-B)\eta - B(3x+y) - 3x - y}{(3x+y)}\right] - \frac{(A-B)^2\mu\eta^2}{4(3x+y)^2}\right|\right]|c_1|^2 \quad (53)$$

Subcase I(a) : when , $\mu \leq \frac{2}{3}\left[\frac{(A-B)\eta - B(3x+y) - 3x - y}{\eta(A-B)(4x+y)}\right](3x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\eta}{6(4x+y)}\left[\frac{(A-B)\eta - B(3x+y)}{(3x+y)}\right] - \frac{(A-B)^2\mu\eta^2}{4(3x+y)^2} \quad (54)$$

Subcase I(b) : when , $\mu \geq \frac{2}{3}\left[\frac{(A-B)\eta - B(3x+y) - 3x - y}{\eta(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \leq \frac{(A-B)\eta}{6(4x+y)}$(55)

Case II: when ,$\mu \geq \frac{2}{3}\left[\frac{(A-B)\eta - B(3x+y)}{\eta(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \leq \frac{(A-B)\eta}{6(4x+y)} + \left[\left|\frac{(A-B)^2\mu\eta^2}{4(3x+y)^2} - \frac{(A-B)\eta}{6(4x+y)}\left\{\frac{(A-B)\eta - B(3x+y)+3x+y}{(3x+y)}\right\}\right|\right]$(56)

Subcase II(a) : when , $\mu \leq \frac{2}{3}\left[\frac{(A-B)\eta - B(3x+y)+3x+y}{\eta(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \leq \frac{(A-B)\eta}{6(4x+y)}$(57)

Subcase II(b) : when , $\mu \geq \frac{2}{3}\left[\frac{(A-B)\eta - B(3x+y)+3x+y}{\eta(A-B)(4x+y)}\right](3x+y)$

$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2\mu\eta^2}{4(3x+y)^2} - \frac{(A-B)\eta}{6(4x+y)}\left[\frac{(A-B)\eta - B(3x+y)}{(3x+y)}\right]$(58)

Combining subcase II(a) and subcase I(b), we get

$|a_3 - \mu a_2^2| \leq \frac{(A-B)\eta}{6(4x+y)}$(59)

iff,

$$\frac{2(3x+y)[(A-B)\eta - B(3x+y) - 3x - y]}{3\eta(4x+y)(A-B)} \leq \mu \leq \frac{2(3x+y)[(A-B)\eta - B(3x+y) + 3x + y]}{3\eta(4x+y)(A-B)}$$

Extremal function

Extreme value for first and third function is

$$z[1 + pz]^q \text{(60)}$$

where $p = \frac{3\eta(A-B)(3x+y) - 4\{\eta(A-B) - B(3x+y)\}(3x+y)}{6(4x+y)(3x+y)}$, $q = \frac{3\eta(4x+y)(A-B)}{3\eta(A-B)(3x+y) - 4\{\eta(A-B) - B(3x+y)\}(3x+y)}$

Extreme value for second function is $\frac{z}{(1-z^2)^p}$(61)

where $p = \frac{(A-B)\eta}{6(4x+y)}$

Singularities:

Special cases(59) on when $A \neq B$

i) In the case of $A > 0, B > 0, \eta > 0$ or $A < 0, B < 0, \eta < 0$ then ,this inequality holds good only for A>B.

ii)) In the case of $A > 0, B > 0, \eta < 0$ or $A < 0, B < 0, \eta > 0$then, this inequality holds good only for B >A.

iii)) In the case of $A > 0, B < 0, \eta < 0$ or $A < 0, B > 0, \eta > 0$ then, this inequality does not hold for all values of A and B.

iv)) In the case of A> 0 , $B < 0$, $\eta > 0$ or A< 0 , $B > 0$, $\eta < 0$ then, this inequality holds good for all values of A and B.

3. Concluding Remarks

If we take A = 1 and B = -1 in the result of theorem 2 ,we get the result of theorem 1, therefore our result for the theorem 2 reduces to the result of the theorem1. Hence theorem 2 is the generalization of theorem 1. And the results are sharp and also if we put A = 1 and B = -1 in extremal function of theorem 2, we get the extremal function of theorem 1.

Similarly if we take A = 1 and B = -1in the result of theorem 4 , we get the result of theorem 3, therefore our result for the theorem 4 reduces to the result of the theorem 3. Hence theorem 4 is the generalization of theorem 3.And the results are sharp and also if we put A = 1 and B = -1 in extremal function of theorem 4, we get the extremal function of theorem 3.

The extremal function given by [(46) and (47)] increases as δ increases and decreases as δ decreases respectively and the extremal function given by [(60) and (61)] also increases and decreases as δ increases and decreases respectively. Hence extremal function is an increasing function.

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